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Set families with a forbidden pattern

Ilan Karpas *

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Abstract

A *balanced pattern* of order $2d$ is an element $P \in \{+, -\}^{2d}$, where both signs appear d times. Two sets $A, B \subset [n]$ form a P -pattern, which we denote by $\text{pat}(A, B) = P$, if $A \triangle B = \{j_1, \dots, j_{2d}\}$ with $1 \leq j_1 < \dots < j_{2d} \leq n$ and $\{i \in [2d] : P_i = +\} = \{i \in [2d] : j_i \in A \setminus B\}$. We say $\mathcal{A} \subset \mathcal{P}[n]$ is P -free if $\text{pat}(A, B) \neq P$ for all $A, B \in \mathcal{A}$. We consider the following extremal question: how large can a family $\mathcal{A} \subset \mathcal{P}[n]$ be if \mathcal{A} is P -free?

We prove a number of results on the sizes of such families. In particular, we show that for some fixed $c > 0$, if P is a d -balanced pattern with $d < c \log \log n$ then $|\mathcal{A}| = o(2^n)$. We then give stronger bounds in the cases when (i) P consists of $d +$ signs, followed by $d -$ signs and (ii) P consists of alternating signs. In both cases, if $d = o(\sqrt{n})$ then $|\mathcal{A}| = o(2^n)$. In the case of (i), this is tight.

1 Introduction

A central goal in extremal set theory is to understand how large a set family can be subject to some restriction on the intersections of its elements. Given $\mathcal{L} \subset \mathbb{N} \cup \{0\}$, we say that a set family \mathcal{A} is \mathcal{L} -intersecting if $|A \cap B| \in \mathcal{L}$ for all distinct $A, B \in \mathcal{A}$. Taking $\mathcal{L}_t = \{s \in \mathbb{N} : s \geq t\}$, a fundamental theorem of Erdős, Ko and Rado [6] shows that \mathcal{L}_t -intersecting families $\mathcal{A} \subset \binom{[n]}{k}$ satisfy $|\mathcal{A}| \leq \binom{n-t}{k-t}$, provided $n \geq n_0(k, t)$. Another important theorem due to Frankl and Füredi [8] shows that if $\mathcal{L}_{\ell, \ell'} := \{s < \ell \text{ or } s \geq k - \ell'\}$, then any $\mathcal{L}_{\ell, \ell'}$ -intersecting family $\mathcal{A} \subset \binom{[n]}{k}$ satisfies $|\mathcal{A}| \leq cn^{\max(\ell, \ell')}$, for some constant c depending on k, ℓ and ℓ' . See [2], [3], [7], [9] for an overview of this extensive topic.

Here we are concerned with understanding the effect of restricting the *pattern* formed between elements of a set family. A *difference pattern* or *pattern* of order t is an element $P \in \{+, -\}^t$. Given such a pattern P , let $S_+(P) = \{i \in [t] : P_i = +\} \subset [t]$ and $s_+(P) = |S_+(P)|$. Define $S_-(P)$ and $s_-(P)$ analogously. Two sets $A, B \subset [n]$ form a *difference pattern* P if:

- (i) $A \triangle B = \{j_1, \dots, j_t\}$ with $j_1 < \dots < j_t$, and
- (ii) $\{i \in [t] : P_i = +\} = \{i \in [t] : j_i \in A \setminus B\}$.

We denote this by writing $\text{pat}(A, B) = P$. A family of subsets $\mathcal{A} \subset \mathcal{P}[n]$ is P -free if $\text{pat}(A, B) \neq P$ for all distinct $A, B \in \mathcal{A}$. In this paper we consider the following natural question: given a pattern P , how large can a family $\mathcal{A} \subset \mathcal{P}[n]$ be if it is P -free?

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First note the following simple observation. If $s_+(P) \neq s_-(P)$ then large P -free families exist. Indeed, if $|s_+(P) - s_-(P)| = m > 0$ then the following families are P -free:

$$\mathcal{B}_1 = \{A \subset [n] : |A| \in [0, m-1] \pmod{2m}\}; \quad \mathcal{B}_2 = \{A \subset [n] : |A| \in [m, 2m-1] \pmod{2m}\}.$$

Clearly either $|\mathcal{B}_1| \geq 2^{n-1}$ or $|\mathcal{B}_2| \geq 2^{n-1}$. We will therefore focus on the case when $s_+(P) = s_-(P) = d$. We say that such patterns are d -balanced. For a balanced pattern P it is only possible that $\text{pat}(A, B) = P$ if $|A| = |B|$. Thus, our question on balanced patterns essentially reduces to a question for uniform families. Given $0 \leq k \leq n$, define

$$f(n, k, P) := \max \left\{ |A| : P\text{-free families } \mathcal{A} \subset \binom{[n]}{k} \right\}.$$

Let $f(n, k, d) = \max\{f(n, k, P) : P \text{ is } d\text{-balanced}\}$. We will also write $\delta(n, k, P)$ and $\delta(n, k, d)$ for the corresponding extremal densities, i.e. $\delta(n, k, P) := f(n, k, P) / \binom{n}{k}$, and $\delta(n, k, d) := f(n, k, d) / \binom{n}{k}$. Note also that if $\mathcal{A} \subset \binom{[n]}{k}$ is P -free then the family $\mathcal{A}^c = \{[n] \setminus A : A \in \mathcal{A}\} \subset \binom{[n]}{n-k}$ is also P -free. Therefore $f(n, k, P) = f(n, n-k, P)$ and it suffices to bound $f(n, k, P)$ for $k \leq n/2$.

Our first aim is to prove a density result for d -balanced patterns of small order. That is, we will show that for fixed d , any sequence of integers $\{k_n\}_{n=1}^\infty$ tending to infinity with n with $k_n \leq n/2$ satisfies $\lim_{n \rightarrow \infty} \delta(n, k_n, d) = 0$. The condition that k is not fixed and tends to infinity with n will be crucial. This is different from the case in the Frankl-Füredi Theorem, which tells us that we can take some fixed $k \geq 2d-1$, $\ell = k-d$ and $\ell' = d-1$, and if $\mathcal{A} \subset \binom{[n]}{k}$ with $|\mathcal{A}| = \omega(n^{k-d})$ then there are $A, B \in \mathcal{A}$ with $|A \triangle B| = 2d$, i.e. A and B form a P -pattern for *some* d -balanced pattern P . Indeed, take any fixed $k := k(d)$, and consider the family $\mathcal{A}_0 \subset \binom{[n]}{k}$ given by

$$\mathcal{A}_0 = \left\{ A \subset [n] : \left| A \cap \left(\frac{(i-1)n}{k}, \frac{in}{k} \right] \right| = 1 \text{ for all } i \in [k] \right\}.$$

Then $|\mathcal{A}_0| \geq c_k n^k$ for some absolute constant $c_k > 0$, but it is easily seen that \mathcal{A}_0 does not contain the pattern $++--$. Therefore, there does not exist a density theorem for d -balanced patterns in subsets of $\binom{[n]}{k}$ with fixed k , as in the Frankl-Füredi theorem.

Our first result shows that such a density theorem does hold for k growing with n .

Theorem 1. *Given $d, k, n \in \mathbb{N}$ with $2k \leq n$ and taking $a_d = (8d)^{5d}$ and $c_d = 6d8^{-d}$ we have*

$$\delta(n, k, d) \leq a_d k^{-c_d}.$$

By our discussion above for fixed k we see that Theorem 1 is in a sense a ‘high-dimensional’ result. Also note that Theorem 1 shows there is a constant $c > 0$ with the property that if P is a d -balanced pattern with $d \leq c \log \log n$ and $\mathcal{A} \subset \mathcal{P}[n]$ which is P -free, then $|\mathcal{A}| = o(2^n)$.

Let $\text{IP}(d)$ denote the d -balanced pattern consisting of d plus signs, followed by d minus signs. We refer to these as *interval patterns*. Given the obstruction of $\text{IP}(2)$ above, it is natural to ask for bounds on $f(n, k, \text{IP}(d))$.

Theorem 2. *Given $d, k, n \in \mathbb{N}$ with $2k \leq n$ we have*

$$\delta(n, k, \text{IP}(d)) = O(d^2 k^{-1}).$$

In particular, families $\mathcal{A} \subset \mathcal{P}[n]$ which are $\text{IP}(d)$ -free for all $d = o(\sqrt{n})$ satisfy $|\mathcal{A}| = o(2^n)$. Furthermore, this turns out to be tight – if $d \geq c\sqrt{n}$ then there are $\text{IP}(d)$ -free families $\mathcal{A} \subset \mathcal{P}[n]$ with $|\mathcal{A}| = \Omega_c(2^n)$.

Lastly, we consider the d -balanced pattern $\text{AP}(d)$ consisting of alternating plus and minus signs, e.g. $\text{AP}(2) = + - + -$. We refer to these as *alternating patterns*. Our next result proves a density result for such patterns.

Theorem 3. *Given $d, k, n \in \mathbb{N}$ with $2k \leq n$ we have*

$$\delta(n, k, \text{AP}(d)) = O\left(\log^{-1}\left(\frac{k}{d^2}\right)\right).$$

Thus again, all families $\mathcal{A} \subset \mathcal{P}[n]$ which are $\text{AP}(d)$ -free for $d = o(\sqrt{n})$ satisfy $|\mathcal{A}| = o(2^n)$. Unlike in the case of the interval patterns, we do not know if this is tight.

Before closing the introduction, we mention some further results related to this topic. A family $\mathcal{A} \subset \mathcal{P}[n]$ is said to be a *tilted Sperner family* if for all distinct $A, B \in \mathcal{A}$ we have $|B \setminus A| \neq 2|A \setminus B|$. Equivalently, \mathcal{A} is P -free for all patterns P with $|S_-(P)| = 2|S_+(P)|$. Kalai raised the question of how large a tilted Sperner family $\mathcal{A} \subset \mathcal{P}[n]$ can be. In [13], Leader and the second author proved that such families satisfy $|\mathcal{A}| \leq (1 + o(1))\binom{n}{n/2}$, which is asymptotically optimal. For sufficiently large n , the extremal families were also determined. In [14], the second author proved that this bound almost still applies if we only forbid ‘tilted pairs’ A, B with a single pattern. It was shown that if $\mathcal{A} \subset \mathcal{P}[n]$ does not contain $A, B \in \mathcal{P}[n]$ with $|B \setminus A| \neq 2|A \setminus B|$ for all distinct $A, B \in \mathcal{A}$ and satisfying $a < b$ for all $a \in A \setminus B$ and $b \in B \setminus A$ then $|\mathcal{A}| \leq C\sqrt{\log n}\binom{n}{n/2}$, for some constant $C > 0$. This condition is equivalent to \mathcal{A} being $P(d)$ -free for all patterns $P(d)$ consisting of $d +$ signs followed by $2d -$ signs. This bound was recently improved by Gerbner and Vizer in [11]. They proved that such families satisfy $|\mathcal{A}| \leq C\sqrt{\log n}\binom{n}{n/2}$. No family is known for this problem with order more than $C\binom{n}{n/2}$.

Lastly, we mention a fascinating question raised by Johnson and Talbot [12] related to Theorem 1 (similar conjectures have been raised by Bollobás, Leader and Malvenuto [4], and Bukh [5]). Our phrasing slightly differs from that in [12].

Question (Johnson–Talbot). *Is it true that for any $k \in \mathbb{N}$ and $\alpha > 0$ there is $n_0(k, \alpha) \in \mathbb{N}$ with the following property. Suppose that $n \geq n_0(k, \alpha)$ and that $\mathcal{A} \subset \binom{[n]}{n/2}$ with $|\mathcal{A}| \geq \alpha\binom{n}{n/2}$. Then there are disjoint sets $S \in \binom{[n]}{n/2 - \lfloor k/2 \rfloor}$ and $T \in \binom{[n] \setminus S}{k}$ such that the family $\mathcal{C}_{T,S} := \{S \cup U : U \in \binom{T}{\lfloor k/2 \rfloor}\}$ is contained in \mathcal{A} .*

This is true for $k = 3$, but is already open for $k = 4$. In this case it is possible to guarantee that $|\mathcal{C}_{T,S} \cap \mathcal{A}| \geq 5$ for some T, S (note $|\mathcal{C}_{T,S}| = 6$ for $k = 4$). More generally, Johnson and Talbot [12] proved that under the hypothesis above, $|\mathcal{C}_{T,S} \cap \mathcal{A}| \geq 4 \cdot 3^{(k-4)/3}$ for some T, S . We note the conclusion that dense subsets of $\mathcal{P}[n]$ contain all small patterns (from Theorem 1) would immediately follow from a positive answer to this question. Indeed, for $k = 2d$ any set $\mathcal{C}_{T,S}$ contains every d -balanced pattern. Theorem 1 may be seen as giving (weak) evidence for the question: for $k = 2d$ and any d -balanced pattern P , there is T and S and sets $A, B \in \mathcal{C}_{T,S} \cap \mathcal{A}$ with $\text{pat}(A, B) = P$.

Notation: Given a set X , we write $\mathcal{P}(X)$ for the power set of X and $\binom{X}{k} = \{A \subset X : |A| = k\}$. Given integers $m, n \in \mathbb{N}$ with $m \leq n$, we let $[n] = \{1, \dots, n\}$ and $[m, n] = \{m, m+1, \dots, n\}$. We also write $(n)_m$ for the falling factorial $(n)_m = n(n-1) \cdots (n-m+1)$.

2 Small balanced patterns

In this section we prove Theorem 1. We will find it convenient to prove many of our results restricted of the middle layer. We then simply write $f(k, P)$ for $f(2k, k, P)$, $\delta(k, P)$ for $\delta(2k, k, P)$, etc.. The following simple observation is useful to move results between different layers of the cube.

Proposition 2.1. *Let $n, m, k, l \in \mathbb{N}$ with $m \leq n$, $l \leq k$ and $k + m - l \leq n$. Let P be a pattern. Then $\delta(n, k, P) \leq \delta(m, l, P)$.*

Proof. Suppose $\mathcal{A} \subset \binom{[n]}{k}$ is P -free with $|\mathcal{A}| = \delta(n, k, P) \binom{n}{k}$. Select two disjoint sets T and U of order m and $k - l$ uniformly at random (possible as $k + m - l \leq n$). Then let $\mathcal{A}_{T,U} = \{A \in \binom{T}{l} : A \cup U \in \mathcal{A}\}$. As \mathcal{A} is P -free, the set $\mathcal{A}_{T,U}$ must also be P -free for all T, U , giving $|\mathcal{A}_{T,U}| \leq \delta(m, l, P) \binom{m}{l}$. However, $\mathbb{E}_{T,U} |\mathcal{A}_{T,U}| = \delta(n, k, P) \binom{m}{l}$. The result follows. \square

Our next two lemmas are the main steps in the proof of Theorem 1. Combined they will allow a recursive bound for $\delta(k, d)$ based on bounds on $\delta(k', d')$ for $k' < k$ and $d' < d$.

Lemma 2.2. *Let $d, k \in \mathbb{N}$ with $k^{1/2} \geq 16 \log k$ and let P be a d -balanced pattern with $P_1 \neq P_{2d}$. Then given any $\gamma \in [\frac{16 \log k}{k^{1/2}}, 1]$ we have*

$$\delta(k, P) \leq \max \left(\gamma, 6\sqrt{\delta(\lceil \gamma^2 k / 64 \rceil, d-1)} \right).$$

Proof. Let γ be chosen as above and let $\mathcal{A} \subset \binom{[2k]}{k}$ be P -free with $|\mathcal{A}| = \alpha \binom{2k}{k}$. If $\alpha \leq \gamma$ then there is nothing to prove, so we will assume that $\alpha > \gamma \geq \frac{16 \log k}{k^{1/2}}$. We will first show that there are many pairs $A, B \in \mathcal{A}$ with $|A \triangle B| = 2$. Indeed, given $C \in \binom{[2k]}{k+1}$ let y_C denote the number of $A \in \mathcal{A}$ with $A \subset C$. Then we have

$$\sum_{C \in \binom{[2k]}{k+1}} y_C = |\{(A, C) \in \mathcal{A} \times \binom{[2k]}{k+1} : A \subset C\}| = |\mathcal{A}|k \geq \alpha k \binom{2k}{k+1}.$$

As for every pair $A, B \in \mathcal{A}$ with $|A \triangle B| = 2$ there is a unique set $C \in \binom{[2k]}{k+1}$ with $A, B \subset C$, we obtain

$$\begin{aligned} \left| \left\{ (A, B) \in \binom{\mathcal{A}}{2} : |A \triangle B| = 2 \right\} \right| &= \sum_{C \in \binom{[2k]}{k+1}} \binom{y_C}{2} \geq \binom{2k}{k+1} \binom{\alpha k}{2} \\ &\geq \frac{\alpha^2 k^2}{4} \times \frac{2k(2k-1)}{(k+1)k} \binom{2k-2}{k-1} \geq \frac{\alpha^2 k^2}{2} \binom{2k-2}{k-1}. \end{aligned} \quad (1)$$

The first inequality holds by the convexity of $\binom{x}{2}$ and the second since $\alpha k - 1 \geq \alpha k / 2$ as $\alpha \geq 2/k$.

Now given $1 \leq i < j \leq 2k$, let $\mathcal{A}_{i,j} := \{A \in \mathcal{A} : |A \cap \{i, j\}| = 2\}$. Note that from (1) we have

$$\sum_{i < j} |\mathcal{A}_{i,j}| = \left| \{(A, B) \in \mathcal{A} \times \mathcal{A} : |A \triangle B| = 2\} \right| \geq \frac{\alpha^2 k^2}{2} \binom{2k-2}{k-1}. \quad (2)$$

Also let $\alpha_{i,j}$ and $\beta_{i,j}$ be defined so that $|\mathcal{A}_{i,j}| = \alpha_{i,j} \binom{2k-2}{k-1}$ and $\beta_{i,j} = (j-i)/2k$. By (2) we find $\{i,j\}$ with $\alpha_{i,j} \geq \frac{\alpha^2}{8}$ and $\beta_{i,j} \geq \frac{\alpha^2}{16}$. Indeed, we have

$$\sum_{\{i,j\}:\alpha_{i,j} < \frac{\alpha^2}{8}} |\mathcal{A}_{i,j}| + \sum_{\{i,j\}:\beta_{i,j} < \frac{\alpha^2}{16}} |\mathcal{A}_{i,j}| < \binom{2k}{2} \frac{\alpha^2}{8} \binom{2k-2}{k-1} + 2k \times \frac{\alpha^2}{16} 2k \binom{2k-2}{k-1} \leq \frac{\alpha^2 k^2}{2} \binom{2k-2}{k-1}.$$

Combined with (2) we see that a claimed pair $\{i,j\}$ exists. Fix such a pair $\{i,j\}$ and set $\mathcal{B} = \mathcal{A}_{i,j}$. Now let $X = [i+1, j-1]$ and $Y = [n] \setminus [i,j]$ so that $\mathcal{B} \subset \binom{X \cup Y}{k-1}$. Partition elements from $\binom{X \cup Y}{k-1}$ according to how they intersect X , for each $\ell \in [0, j-i-2]$ letting

$$X_\ell = \left\{ A \in \binom{X \cup Y}{k-1} : |A \cap X| = \ell \right\}.$$

Also let $\mathcal{B}_\ell = \mathcal{B} \cap X_\ell$ and $L = \left\{ \ell : \left| \ell - \frac{|X|}{2} \right| \leq \sqrt{|X| \log \left(\frac{8}{\alpha} \right)} \right\}$. By Chernoff's inequality we have

$$\sum_{\ell \notin L} |X_\ell| \leq \frac{\alpha^2}{32} \binom{|X| + |Y|}{k-1}.$$

Using that $|\mathcal{B}| = \alpha_{i,j} \binom{|X|+|Y|}{k-1} \geq \frac{\alpha^2}{16} \binom{|X|+|Y|}{k-1}$ this shows that

$$\sum_{\ell \in L} |\mathcal{B}_\ell| \geq |\mathcal{B}| - \frac{\alpha^2}{32} \binom{|X| + |Y|}{k-1} \geq \frac{\alpha^2}{32} \binom{|X| + |Y|}{k-1} \geq \frac{\alpha^2}{32} \sum_{\ell \in L} |X_\ell|.$$

The last inequality here holds since the sets X_ℓ are disjoint subsets of $\binom{X \cup Y}{k-1}$. Thus for some $\ell \in L$ we have $|\mathcal{B}_\ell| \geq \frac{\alpha^2}{32} |X_\ell|$. By averaging, we find a set $U \subset Y$ with $|U| = k - \ell - 1$ such that the family $\mathcal{C} = \{C \in \binom{X}{\ell} : C \cup U \in \mathcal{B}_\ell\}$ satisfies $|\mathcal{C}| \geq \frac{\alpha^2}{32} \binom{|X|}{\ell}$.

To complete the proof, let Q denote the pattern obtained from P by removing P_1 and P_{2d} , i.e. $Q = P_2 \cdots P_{2d-1}$. Note that as $P_1 \neq P_{2d}$ we see that Q is $(d-1)$ -balanced. We claim that \mathcal{C} is Q -free. Indeed, suppose $C_1, C_2 \in \mathcal{C}$ with $\text{pat}(C_1, C_2) = Q$. Then by definition of \mathcal{C} and $\mathcal{B} = \mathcal{A}_{i,j}$ we have

$$\left\{ C_a \cup U \cup \{h\} : a \in \{1, 2\}, h \in \{i, j\} \right\} \subset \mathcal{A}.$$

If $P_1 = +$ we find $\text{pat}(C_1 \cup U \cup \{i\}, C_2 \cup U \cup \{j\}) = P$. If $P_1 = -$ we find $\text{pat}(C_1 \cup U \cup \{j\}, C_2 \cup U \cup \{i\}) = P$. Thus \mathcal{C} must be Q -free and

$$\frac{\alpha^2}{32} \binom{|X|}{\ell} \leq |\mathcal{C}| \leq \delta(|X|, \ell, Q) \binom{|X|}{\ell}.$$

Take $k' = \lfloor \frac{|X|}{2} - \sqrt{|X| \log \left(\frac{8}{\alpha} \right)} \rfloor$. A calculation shows that $\frac{|X|}{4} \geq \sqrt{|X| \log \left(\frac{8}{\alpha} \right)} + 2$ since $\alpha \geq \frac{16 \log k}{k^{1/2}}$ and $|X| + 2 = \beta_{i,j} 2k \geq \frac{\alpha^2 k}{8}$. This gives

$$k' \geq \frac{|X|}{2} - \sqrt{|X| \log \left(\frac{8}{\alpha} \right)} - 1 \geq \frac{|X|}{4} + 1 \geq \left\lceil \frac{\beta_{i,j} k}{2} \right\rceil \geq \left\lceil \frac{\alpha^2 k}{64} \right\rceil \geq \left\lceil \frac{\gamma^2 k}{64} \right\rceil.$$

Since $\ell \in L$ we have $k' \leq \ell \leq |X| - k'$. Using Proposition 2.1 we find that $\frac{\alpha^2}{32} \leq \delta(|X|, \ell, d-1) \leq \delta(2k', k', d-1) = \delta(k', d-1) \leq \delta(\lceil \frac{\gamma^2 k}{64} \rceil, d-1)$. Rearranging this gives $\alpha \leq 6 \sqrt{\delta(\lceil \frac{\gamma^2 k}{64} \rceil, d-1)}$. \square

Our second lemma deals with the case where P starts and ends with the same signs.

Lemma 2.3. *Let $d \in \mathbb{N}$ and let P be a d -balanced pattern with $P_1 = P_{2d}$. Then there are $d_1, d_2 \geq 1$ with $d_1 + d_2 = d$ such that the following holds. For every k_1, k_2 with $2k_1 + k_2 = k$ we have*

$$\delta(k, P) \leq \max \left(2e^{-k_1/12}, 4\delta(k_1, d_1), 4(3k_1)^{2d_1} \delta(k_2, d_2) \right).$$

Similarly for every k_1, k_2 with $k_1 + 2k_2 = k$ we have

$$\delta(k, P) \leq \max \left(2e^{-k_2/12}, 4\delta(k_2, d_2), 4(3k_2)^{2d_2} \delta(k_1, d_1) \right).$$

Proof. To begin, for each $\ell \in [0, 2d]$ let

$$c_\ell = |\{j \in [\ell] : P_j = +\}| - |\{j \in [\ell] : P_j = -\}|.$$

As P is d -balanced and $P_1 = P_{2d}$, we have $c_{2d-1} = -c_1$. Combined with the fact that $c_0 = c_{2d} = 0$ and c_ℓ changes by exactly 1 as ℓ increases, we see that $c_{2d_1} = 0$ for some $1 \leq d_1 \leq d-1$. Setting $d_2 := d - d_1$ and $Q_1 = P_1 \cdots P_{2d_1}$, $Q_2 = P_{2d_1+1} \cdots P_{2d}$ it is easy to see that these patterns are d_1 -balanced and d_2 -balanced respectively.

Now suppose that $\mathcal{A} \subset \binom{[2k]}{k}$ with $|\mathcal{A}| = \alpha \binom{2k}{k}$ and that \mathcal{A} is P -free. We will prove the first bound above as the second bound is proved identically. We will assume that $\alpha \geq 2e^{-k_1/12}$ as otherwise there is nothing to show. Partition $[2k]$ into two consecutive intervals $I_1 = [3k_1]$ and $I_2 = [3k_1 + 1, 2k]$. For each $\ell \in I_1$ let $Z_\ell := \binom{I_1}{\ell} \times \binom{I_2}{k-\ell}$. Let $L = \left\{ \ell \in I_1 : |\ell - 3k_1/2| \leq \sqrt{3k_1 \log(\frac{2}{\alpha})} \right\}$. Note that as $|\bigcup_{\ell \notin L} Z_\ell| \leq \frac{\alpha}{2} \binom{2k}{k}$ by Chernoff's inequality, we have $|\mathcal{A} \cap Z_\ell| \geq \frac{\alpha}{2} |Z_\ell|$ for some $\ell \in L$. Fix such a choice of ℓ and set $Z := Z_\ell$ and $\mathcal{B} = \mathcal{A} \cap Z_\ell$ so that $\mathcal{B} \subset Z$ with $|\mathcal{B}| \geq \frac{\alpha}{2} |Z|$.

We will now prove that α satisfies

$$\alpha \leq \max \left(4\delta(|I_1|, \ell, Q_1), 4|I_1|^{2d_1} \delta(|I_2|, k - \ell, Q_2) \right). \quad (3)$$

To see this, we may assume that $\alpha \geq 4\delta(|I_1|, \ell, Q_1)$ as otherwise there is nothing to show. Consider the set \mathcal{P}_{Q_1} given by

$$\mathcal{P}_{Q_1} = \left\{ (A, B) \in Z \times Z : \text{pat}(A \cap I_1, B \cap I_1) = Q_1 \text{ and } A \cap I_2 = B \cap I_2 \right\}.$$

We will first show that $|\mathcal{B} \times \mathcal{B}| \cap \mathcal{P}_{Q_1} \geq \frac{\alpha}{4|I_1|^{2d_1}} |\mathcal{P}_{Q_1}|$. Indeed, for each $D \in \binom{I_2}{k-\ell}$ let

$$\mathcal{E}(D) := \left\{ C \in \binom{I_1}{\ell} : C \cup D \in \mathcal{B} \right\}; \quad \mathcal{P}_{Q_1}(D) := \{C, C' \in \mathcal{E}(D) : \text{pat}(C, C') = Q_1\}.$$

Noting that each $\mathcal{C} \subset \binom{I_1}{\ell}$ with $|\mathcal{C}| > \delta(|I_1|, \ell, Q_1) \binom{|I_1|}{\ell}$ contains C, C' with $\text{pat}(C, C') = Q_1$, we find $|\mathcal{P}_{Q_1}(D)| \geq |\mathcal{E}(D)| - \delta(|I_1|, \ell, Q_1) \binom{|I_1|}{\ell}$. Combined these give

$$|\mathcal{B} \times \mathcal{B}| \cap \mathcal{P}_{Q_1} = \sum_{D \in \binom{I_1}{\ell}} |\mathcal{P}_{Q_1}(D)| \geq \sum_{D \in \binom{I_2}{k-\ell}} \left(|\mathcal{E}(D)| - \delta(|I_1|, \ell, Q_1) \binom{|I_1|}{\ell} \right) \geq \frac{\alpha}{4} |Z|, \quad (4)$$

The final inequality here holds since $\sum_{D \in \binom{I_2}{k-\ell}} |\mathcal{E}(D)| = |\mathcal{B}| \geq \frac{\alpha}{2} |Z|$ and $\alpha \geq 4\delta(|I_1|, \ell, Q_1)$. Lastly, using that $|\mathcal{P}_{Q_1}| \leq |I_1|^{2d_1} |Z|$ together with (4), we obtain $|(\mathcal{B} \times \mathcal{B}) \cap \mathcal{P}_{Q_1}| \geq \frac{\alpha}{4|I_1|^{2d_1}} |\mathcal{P}_{Q_1}|$.

Now, from this bound we find a choice of $C, C' \in \binom{I_1}{\ell}$ with $\text{pat}(C, C') = Q_1$ such that the set

$$\mathcal{F}_{C, C'} = \left\{ D \in \binom{I_2}{k-\ell} : C \cup D, C' \cup D \in \mathcal{A} \right\}$$

satisfies $|\mathcal{F}_{C, C'}| \geq \frac{\alpha}{4|I_1|^{2d_1}} \binom{n_2}{k-\ell}$. However, if $D, D' \in \mathcal{F}$ with $\text{pat}(D, D') = Q_2$ then $C \cup D, C' \cup D' \in \mathcal{A}$ and $\text{pat}(C \cup D, C' \cup D') = Q_1 Q_2 = P$. As \mathcal{A} is P -free we see $\mathcal{F}_{C, C'} \subset \binom{I_2}{k-\ell}$ is Q_2 -free. This gives $\frac{\alpha}{4|I_1|^{d_1}} \leq \delta(|I_2|, k-\ell, Q_2)$ and proves (3).

To complete the proof, note that as $\alpha \geq 2e^{-k_1/12}$, by definition of L we have $\ell \in L \subset [k_1, 2k_1]$ and $k-\ell \in [k_2, k_2+k_1]$. As $|I_1| = 3k_1$ and $|I_2| = 2k - 3k_1 = 2k_2 + k_1$, by Proposition 2.1 we find

$$\delta(|I_1|, \ell, Q_1) \leq \delta(2k_1, k_1, Q_1) = \delta(k_1, d_1) \quad \text{and} \quad \delta(|I_2|, k-\ell, Q_1) \leq \delta(2k_2, k_2, Q_2) = \delta(k_2, d_2).$$

Combined with (3) this completes the proof. \square

Proof of Theorem 1. We prove by induction on d that with $a_d = (8d)^{5d}$ and $c_d = 6d8^{-d}$ we have

$$\delta(k, d) \leq a_d k^{-c_d}. \quad (5)$$

For $d = 1$ we have $P = +- \text{ or } P = -+$ and $\mathcal{A} \subset \binom{[2k]}{k}$ is P -free simply means that $|A \triangle B| \neq 2$ for all distinct $A, B \in \mathcal{A}$. It is well known that such families satisfy $|\mathcal{A}| \leq \frac{1}{k} \binom{2k}{k}$. Indeed, for each $C \in \binom{[2k]}{k+1}$ let y_C denote the number of $A \in \mathcal{A}$ with $A \subset C$. Then

$$\sum_{C \in \binom{[2k]}{k+1}} y_C = |\{(A, C) \in \mathcal{A} \times \binom{[2k]}{k+1} : A \subset C\}| = |\mathcal{A}| \times k.$$

However, if $|A \triangle B| \neq 2$ for all distinct $A, B \in \mathcal{A}$ we must have $y_C \leq 1$ for all C . Rearranging, we obtain the claimed upper bound on $|\mathcal{A}|$. This easily gives that (5) holds for $d = 1$.

We now prove the result for a d -balanced pattern P , assuming by induction that the theorem holds for all d' -balanced patterns with $d' < d$. We can assume that $k \geq a_d^{1/c_d} \geq 16^8$ as otherwise the statement is trivial. We will first prove this when P begins and ends with different signs, using Lemma 2.2, noting that in this range $k^{1/2} \geq 16 \log k$. To apply this, let $\gamma = 8(a_{d-1})^{1/2} k^{-\frac{c_{d-1}}{4}}$ and note that $\gamma \geq 8(a_{d-1})^{1/2} k^{-\frac{1}{4}} \geq 16(\log k) k^{-1/2}$ since $k^{1/4}/\log k \geq 1/32 \geq 2(a_{d-1})^{-1/2}$. Therefore we can apply Lemma 2.2 to find

$$\begin{aligned} \delta(k, P) &\leq \max\left(\gamma, 6\sqrt{\delta\left(\left\lceil \frac{\gamma^2 k}{64} \right\rceil, d-1\right)}\right) \leq \max\left(8(a_{d-1})^{1/2} k^{-\frac{c_{d-1}}{4}}, 6\sqrt{a_{d-1}(a_{d-1} k^{1-\frac{c_{d-1}}{2}})^{-c_{d-1}}}\right) \\ &\leq 8(a_{d-1})^{1/2} k^{-\frac{c_{d-1}}{4}} \leq a_d k^{-c_d}. \end{aligned}$$

The second inequality here uses that Lemma 2.2 holds for $d-1$ by induction, the third that $(a_{d-1})^{-c_{d-1}} \leq 1$ and $1 - \frac{c_{d-1}}{2} \geq \frac{1}{2}$ and the last inequality uses that $c_d \leq \frac{c_{d-1}}{4}$.

We now move to the case where P starts and ends with the same signs. Given P let d_1 and d_2 be as in Lemma 2.3 so that $d_1 + d_2 = d$ with $d_i \geq 1$. We will assume that $d_1 \leq d_2$ as the other case follows

similarly. Let us set $k_1 = \lceil k^\beta \rceil$ where $\beta = \frac{c_{d_2}}{2d_1 + c_{d_1}}$. Set $k_2 = k - 2k_1 \geq k - 4k^\beta \geq k - 4k^{1/2} \geq \frac{k}{2}$ for $k \geq 2^6$. Then by Lemma 2.3 we have

$$\begin{aligned} \delta(k, P) &\leq \max \left(2e^{-k_1/12}, 4\delta(k_1, d_1), 4(3k_1)^{2d_1} \delta(k_2, d_2) \right) \\ &\leq \max \left(2e^{-k^\beta/12}, 4a_{d_1} r^{-\beta c_{d_1}}, 4(6k^\beta)^{2d_1} a_{d_2} \left(\frac{k}{2}\right)^{-c_{d_2}} \right) \\ &\leq \max \left(2e^{-k^\beta/12}, 4a_{d_1} k^{-\beta c_{d_1}}, 8^{2d_1+3} a_{d_2} k^{2d_1\beta - c_{d_2}} \right) \\ &\leq \max \left(2e^{-k^\beta/12}, 8^{2d_1+3} a_{d_2} k^{-\frac{c_{d_1} c_{d_2}}{2d_1 + c_{d_1}}} \right) \leq a_d k^{-c_d}. \end{aligned}$$

The first part of the final inequality here uses $a_d \geq 2k^{c_d}$ for $k \leq (a_d/2)^{1/c_d}$ and that $e^{-k^\beta/12} \leq k^{-c_d}$ for $k \geq (a_d/2)^{1/c_d}$. The second part uses that $8^{2d_1+3} a_{d_2} \leq a_d$ and that since $d = d_1 + d_2$ and $d \leq 2d_2$ we have $c_d \leq 12d_2 8^{-d} \leq \frac{36d_1 d_2 8^{-(d_1+d_2)}}{2d_1+1} \leq \frac{c_{d_1} c_{d_2}}{2d_1+1} \leq \frac{c_{d_1} c_{d_2}}{2d_1+c_{d_1}}$. This completes this case and the proof of the theorem. \square

3 Interval patterns

In this section, we first prove Theorem 2. We then give several lower bounds for the case $n = 2k$ depending on value of d .

3.1 Upper Bound on $\delta(n, n/2, \text{IP}(d))$

Proof of Theorem 2. Let $m = \lfloor \frac{n}{8d^2} \rfloor$. We partition $[n]$ into m intervals, $[n] = I_1 \cup \dots \cup I_m$ with $|I_i| = \lfloor 8d^2 \rfloor$ or $|I_i| = \lceil 8d^2 \rceil$ for all $i \in [m]$.

Consider the following way of choosing elements from $\binom{[n]}{n/2}$. First select a set $T \subset \binom{[n]}{n/2-d}$ uniformly at random. Let $J = \{i \in [m] : |I_i \setminus T| \geq d\}$. As $d < |I_i|/2$, for every $i \in [m]$ we have

$$\mathbb{P}(i \in J) = \mathbb{P}(|I_i \setminus T| \geq d) > \mathbb{P}(|I_i \cap T| \leq |I_i|/2) \geq \frac{1}{2}.$$

If $i \in J$, further select a set $S_i \subset \binom{I_i \setminus T}{d}$ uniformly at random, and set $A_i = T \cup S_i$. If $i \in [m] \setminus J$ simply set $A_i = \emptyset$.

Now for every $i, j \in J$ with $i < j$, we have $\text{pat}(A_i, A_j) = \text{IP}(d)$. Also for $i \notin J$ we have $A_i \notin \mathcal{A}$, since $|A_i| = 0 \neq n/2$. We conclude that there is at most one index $i \in [m]$ with $A_i \in \mathcal{A}$. Equivalently,

$$\sum_{i=1}^m \mathbf{1}_{A_i \in \mathcal{A}} = \sum_{i=1}^m \sum_{A \in \mathcal{A}} \mathbf{1}_{A_i = A} \leq 1.$$

This is true for any choice of T and S_i 's, so in particular if we take the expectation on both sides, we have

$$\sum_{i=1}^m \sum_{A \in \mathcal{A}} \mathbb{P}(A_i = A) \leq 1. \tag{6}$$

But as $A_i \notin \mathcal{A}$ for $i \notin J$, given any $A \in \mathcal{A}$ we get that $\mathbb{P}(A_i = A) = \mathbb{P}(A_i = A | i \in J) \mathbb{P}(i \in J) > \frac{1}{2} \mathbb{P}(A_i = A | i \in J)$. Rewriting (6), this gives

$$\sum_{i=1}^m \sum_{A \in \mathcal{A}} \frac{\mathbb{P}(A_i = A | i \in J)}{2} \leq 1. \quad (7)$$

Lemma 3.1. *Let $A \in \binom{[n]}{n/2}$ be a fixed set. If $|A \cap I_i| \geq \frac{|I_i|}{2} + d$, then $\mathbb{P}(A_i = A | i \in J) \geq \frac{1}{\binom{n}{n/2}}$.*

Proof. Indeed, $\mathbb{P}(A_i = A | i \in J) = \frac{N_i(A)}{N_i}$ where

$$N_i(A) := \left| \left\{ (S_i, T) : S_i \in \binom{I_i}{d}, T \in \binom{[n] \setminus S_i}{n/2 - d}, S_i \cup T = A \right\} \right|;$$

$$N_i := \left| \left\{ (S_i, T) : S_i \in \binom{I_i}{d}, T \in \binom{[n] \setminus S_i}{n/2 - d} \right\} \right|.$$

However, we have

$$\frac{N_i(A)}{N_i} \geq \frac{\binom{4d^2+d}{d}}{\binom{8d^2}{d} \binom{n-d}{\frac{n}{2}-d}} = \frac{(4d^2+d)d \left(\frac{n}{2}-d\right)! \frac{n!}{2!}}{(8d^2)_d (n-d)!} \geq \frac{\left(\frac{n}{2}-d\right)! \frac{n!}{2!}}{2^d (n-d)!} > \frac{(n/2)!(n/2)!}{(n)!} = \frac{1}{\binom{n}{n/2}}.$$

□

For a set $A \in \binom{[n]}{n/2}$, denote by $G(A) = \left| \left\{ i \in [m] : |A \cap I_i| \geq \frac{|I_i|}{2} + d \right\} \right|$. From Lemma 3.1 it follows that for any given A , we have $\sum_{i=1}^m \mathbb{P}(A_i = A | i \in J) \geq G(A) \times \frac{1}{\binom{n}{n/2}}$. Together with (7), we obtain

$$\sum_{A \in \mathcal{A}} G(A) \leq 2 \binom{n}{n/2}. \quad (8)$$

We call a set $A \in \binom{[n]}{n/2}$ *bad*, if $G(A) < m/5$. Otherwise, we say that A is *good*. Let \mathcal{B} be the family of all bad sets.

Lemma 3.2. $|\mathcal{B}| = o\left(\frac{1}{m} \binom{n}{n/2}\right)$ for sufficiently large n .

Proof. For a uniform random choice of a set $A \subseteq \binom{[n]}{n/2}$, let X_i be a random variable, with $X_i = 1$ if $|A \cap I_i| > \frac{|I_i|}{2} + d$, and $X_i = 0$ otherwise. Let $Z = X_1 + \dots + X_m$. To prove the lemma, we need to show that $\mathbb{P}(Z < m/5) = o\left(\frac{1}{m}\right)$. By linearity of expectation, $\mathbb{E}Z = m\mathbb{E}X_i = m\mathbb{P}(X_i = 1)$. Notice that for every $i \neq j$, X_i and X_j are negatively correlated, since if A has many elements in one interval, it is less likely to have many elements on another interval.

$$\mathbb{P}(X_i = 0) = \frac{\sum_{i=0}^{4d^2+d} \binom{8d^2}{i} \binom{n-8d^2}{n/2-i}}{\binom{n}{n/2}} \leq \frac{1}{2} + \frac{\sum_{i=4d^2}^{4d^2+d} \binom{8d^2}{i} \binom{n-8d^2}{n/2-i}}{\binom{n}{n/2}} \leq \frac{1}{2} + \frac{d \binom{8d^2}{4d^2} \binom{n-8d^2}{n/2-4d^2}}{\binom{n}{n/2}} < 0.79.$$

The second inequality uses Stirling's formula. Therefore $\mathbb{P}(X_i = 1) = \mathbb{E}X_i > 0.21$. Using linearity of expectation gives $\mathbb{E}Z = \sum_{i=1}^m \mathbb{E}X_i > 0.21m$.

By a version of the Chernoff-Hoeffding bound for negatively correlated variables [15], we deduce that $\mathbb{P}(A \in \mathcal{B}) = \mathbb{P}(Z < 0.2m) < \mathbb{P}(Z - \mathbb{E}Z > 0.01m) = o\left(\frac{1}{m}\right)$, finishing the proof. □

Therefore, if $|\mathcal{A}| \geq \frac{2}{m} \binom{n}{n/2}$, then $|\mathcal{A} \setminus \mathcal{B}| = (1 - o(1))|\mathcal{A}|$. Using (8), we see that

$$(1 - o(1)) \frac{m|\mathcal{A}|}{10} \leq \sum_{A \in \mathcal{A} \setminus \mathcal{B}} G(A) \leq \sum_{A \in \mathcal{A}} G(A) \leq \binom{n}{n/2}. \quad (9)$$

Equivalently $|\mathcal{A}| = O(\frac{1}{m} \binom{n}{n/2}) = O(\frac{d^2}{n} \binom{n}{n/2})$, as required. \square

3.2 Lower Bound on $\delta(n, n/2, \text{IP}(d))$

For the lower bounds, we provide different lower bounds, depending on the range of d .

Theorem 4. *The following hold:*

(i) *If $d = o(\sqrt{n})$, there is an $\text{IP}(d)$ -free family $\mathcal{A} \subseteq \binom{[n]}{n/2}$ with $|\mathcal{A}| = \Omega(\max\{\frac{1}{nd}, \frac{d^2}{n^{3/2}}\} \cdot \binom{n}{n/2})$.*

(ii) *If $d = c\sqrt{n}$, there is an $\text{IP}(d)$ -free family $\mathcal{A} \subseteq \binom{[n]}{n/2}$ with $|\mathcal{A}| = \Omega_c(\binom{n}{n/2})$.*

Proof. First we prove (i). For a set $A \in \binom{[n]}{n/2}$ let $S(A) := \sum_{i \in A} i$, the sum of the elements in A . Observe that if $\text{pat}(A, B) = \text{IP}(d)$ then $0 < |S(A) - S(B)| < nd$. Thus for any $0 \leq i \leq nd - 1$, the family $\mathcal{A}_i := \{A \in \binom{[n]}{n/2} \mid S(A) \equiv i \pmod{nd}\}$ forms an $\text{IP}(d)$ -free family. By the pigeonhole principle, we can find such i so that $|\mathcal{A}_i| \geq \frac{1}{nd} \binom{n}{n/2}$.

To obtain the second bound from (i), note that if we choose a set $A \in \binom{[n]}{n/2}$ uniformly at random,

$$\mathbb{E}[S(A)] = \frac{n(n+1)}{4}. \quad (10)$$

To calculate the variance, let

$$X_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A. \end{cases}$$

Then $S(A) = \sum_{i=1}^n iX_i$. Now $\mathbb{E}[X_i] = \frac{1}{2}$ every $i \in [n]$ and $\mathbb{E}[X_i X_j] = \frac{1}{4}(1 - \frac{1}{n-1})$. Using this, we find

$$\begin{aligned} \text{Var}(S(A)) &= \mathbb{E}[(\sum_{i=1}^n iX_i)^2] - \mathbb{E}[\sum_{i=1}^n iX_i]^2 \leq \sum_{i \in [n]} i^2 \mathbb{E}[X_i] + \sum_{i \neq j} ij (\mathbb{E}[X_i X_j] - \mathbb{E}[X_i] \mathbb{E}[X_j]) \\ &\leq \sum_{i \in [n]} \frac{i^2}{2} \leq \frac{n^3}{2}. \end{aligned} \quad (11)$$

From (10) and (11) together, by Chebyshev's inequality we get $\mathbb{P}(|S(A) - n(n+1)/4| \leq n^{3/2}) \geq 1/2$. Equivalently, $|\{A \in \binom{[n]}{n/2} : |S(A) - n(n+1)/4| \leq n^{3/2}\}| \geq \frac{1}{2} \binom{n}{n/2}$. By an easy averaging argument, for some value $m \in [\frac{n(n+1)}{4} - \frac{n^{3/2}}{3}, \frac{n(n+1)}{4} + \frac{n^{3/2}}{2}]$.

$$|\{A \in \binom{[n]}{n/2} : S(A) \in [m - \frac{d^2}{2}, m + \frac{d^2}{2}]\}| \geq \frac{1}{2(2n^{3/2}/d^2 + 2)} \binom{n}{n/2} = \Omega\left(\frac{d^2}{n^{3/2}} \binom{n}{n/2}\right)$$

However, since two sets $A, B \in \binom{[n]}{n/2}$ with $\text{pat}(A, B) = \text{IP}(d)$ have $|S(A) - S(B)| > d^2$, this completes the proof of (i).

To prove (ii), let $c > 0$ be given and let $d = c\sqrt{n}$. Note that if $\text{pat}(A, B) = IP(d)$ then for some $i \in [n]$ we have $|A \cap [i]| \geq |B \cap [i]| + d$. This shows that $\mathcal{A} = \{A \in \binom{[n]}{n/2} : ||A \cap [i]| - i/2| < d/4 \text{ for all } i \in [n]\}$ is an $IP(d)$ -free family. We will now show that $|\mathcal{A}| = \Omega_c(\binom{n}{n/2})$.

To see this, it is convenient to identify elements of $\binom{[n]}{n/2}$ with certain walks. Let \mathcal{W}_0 denote the set of all walks $W = W_0 \cdots W_n$ of length n on \mathbb{Z} with $W_0 = W_n = 0$ and which either increase or decrease by 1 in each step (i.e. $|W_i - W_{i-1}| = 1$ for all $i \in [n]$). Note that each walk $W \in \mathcal{W}_0$ naturally corresponds to a subset of $[n]$ of size $n/2$ consisting of those steps in $[n]$ where the walk increases. Under this correspondence, the set \mathcal{A} corresponds to those walks in \mathcal{W}_0 which lie entirely in $[-d/4, d/4]$.

Now select a walk $W \in \mathcal{W}_0$ uniformly at random. Letting T denote a value to be determined, consider the following events:

$$\begin{aligned} A &= \{W_j \in [-d/4, d/4] \text{ for all } j \in [n]\} \\ B &= \{W_{in/T} \in [-d/12, d/12] \text{ for all } i \in [T-1]\} \\ C_i &= \{W_j \in [-d/4, d/4] \text{ for all } j \in [\frac{(i-1)n}{T}, \frac{in}{T}]\}, \text{ where } i \in [T]. \end{aligned}$$

Also for $i \in [T-1]$ and $a_i \in [-d/12, d/12]$, let $B_i(a_i)$ denote the event $B_i(a_i) = \{W_{in/T} = a_i\}$. We will show that

$$\mathbb{P}_{W \sim \mathcal{W}_0} \left(B \wedge \bigwedge_{i \in [T]} C_i \right) \geq c' > 0, \quad (12)$$

where c' depends only on c . Since $\bigwedge_{i \in [T]} C_i \subset A$, this will prove the result.

To begin, note that we have

$$\begin{aligned} \mathbb{P}_{W \sim \mathcal{W}_0} \left(B \wedge \bigwedge_{i \in [T]} C_i \right) &\geq \sum_{a_1, \dots, a_{T-1} \in [-d/12, d/12]} \mathbb{P}_{W \sim \mathcal{W}_0} \left(\bigwedge_{i \in [T-1]} B_i(a_i) \wedge \bigwedge_{i \in [T]} C_i \right) \\ &= \sum_{a_1, \dots, a_{T-1} \in [-d/12, d/12]} \mathbb{P}_{W \sim \mathcal{W}_0} \left(\bigwedge_{i \in [T]} C_i \mid \bigwedge_{i \in [T-1]} B_i(a_i) \right) \\ &\quad \times \mathbb{P}_{W \sim \mathcal{W}_0} \left(\bigwedge_{i \in [T-1]} B_i(a_i) \right). \end{aligned} \quad (13)$$

Let $\mathcal{W}(a, b)$ denote the collection of random walks of length n/T which start at a and end at b . Since C_i depends only on $\{W_j : j \in [(i-1)n/T, in/T]\}$, taking $a_0 = a_T = 0$ we have

$$\begin{aligned} \mathbb{P}_{W \sim \mathcal{W}_0} \left(\bigwedge_{i \in [T]} C_i \mid \bigwedge_{i \in [T-1]} B_i(a_i) \right) &= \prod_{i \in [T]} \mathbb{P}_{W \sim \mathcal{W}_0} \left(C_i \mid B_{i-1}(a_{i-1}) \wedge B_i(a_i) \right) \\ &= \prod_{i \in [T]} \mathbb{P}_{W \sim \mathcal{W}(a_{i-1}, a_i)} \left(W \text{ lies entirely in } [-d/4, d/4] \right). \end{aligned} \quad (14)$$

Claim: For every $a, b \in [-d/12, d/12]$ we have $\mathbb{P}_{W \sim \mathcal{W}(a, b)} \left(W \text{ lies entirely in } [-d/4, d/4] \right) \geq 1/2$.

Let $\mathcal{W}(a)$ denote the collection of all walks of length n/T which begin at a . Let us select W from $\mathcal{W}(a)$ uniformly at random and let $S_{n/T}$ denote the final vertex. By the reflection principle for random

walks, we have

$$\begin{aligned}
\mathbb{P}_{W \sim \mathcal{W}(a,b)}(W \text{ exceeds } d/4) &= \mathbb{P}_{W \sim \mathcal{W}(a)}(W \text{ exceeds } d/4 | S_{n/T} = b) \\
&= \frac{\mathbb{P}_{W \sim \mathcal{W}(a)}(S_{n/T} = d/2 - b)}{\mathbb{P}_{W \sim \mathcal{W}(a)}(S_{n/T} = b)} \\
&= \frac{\binom{n/T}{n/2T + (d/2 - b) - a}}{\binom{n/T}{n/2T + b - a}} \leq \frac{\binom{n/T}{n/2T + d/3}}{\binom{n/T}{n/2T + d/6}} \\
&= \frac{(n/2T - d/6)_{d/6}}{(n/2T + d/6)_{d/6}} \leq \left(1 - \frac{dT}{3n}\right)^{d/6} \leq e^{-d^2 T / 36n}.
\end{aligned}$$

Taking $T = 72/c^2$ say, we find $\mathbb{P}_{W \sim \mathcal{W}(a,b)}(W \text{ exceeds } d/4) \leq e^{-2} < 1/4$. By symmetry, this gives $\mathbb{P}_{W \sim \mathcal{W}(a,b)}(W \text{ lies entirely in } [-d/4, d/4]) \geq 1 - 2 \times (1/4) = 1/2$, as claimed.

Now by combining (14) together with the claim in (13) we find

$$\mathbb{P}_{W \sim \mathcal{W}_0} \left(B \wedge \bigwedge_{i \in [T]} C_i \right) \geq \sum_{a_1, \dots, a_{T-1} \in [-d/12, d/12]} 2^{1-T} \times \mathbb{P}_{W \sim \mathcal{W}_0} \left(\bigwedge_{i \in [T-1]} B_i(a_i) \right). \quad (15)$$

But letting $b_i := \frac{n}{2T} + a_i - a_{i-1}$ for all $i \in [T]$ where $a_0 = a_T = 0$, we have

$$\mathbb{P}_{W \sim \mathcal{W}_0} \left(\bigwedge_{i \in [T-1]} B_i(a_i) \right) = \frac{\prod_{i \in [T]} \binom{n/T}{b_i}}{\binom{n}{n/2}} = \Omega_{c,T}(d^{1-T}).$$

The final inequality follows by Stirling's approximation, using that $b_i \in [\frac{n}{2T} - \frac{d}{6}, \frac{n}{2T} + \frac{d}{6}]$ for all $i \in [T]$. Combined with (15), this gives $\mathbb{P}_{W \sim \mathcal{W}_0} \left(B \wedge \bigwedge_{i \in [T]} C_i \right) = \Omega_{c,T}(1) = \Omega_c(1)$, as required. \square

4 Alternating patterns

To begin, we prove an auxiliary lemma. Given $\mathbf{x} = (x_i)$ and $\mathbf{y} = (y_i)$ in $[m]^D$ we say that \mathbf{y} d -dominates \mathbf{x} if $|\{i \in [D] : x_i \neq y_i\}| = d$ and $x_i \leq y_i$ for all $i \in [D]$.

Lemma 4.1. *Let $d, m, D \in \mathbb{N}$ with $2md^2 \leq D$. Suppose that $\mathcal{C} \subset [m]^D$ does not contain \mathbf{x} and \mathbf{y} such that \mathbf{y} d -dominates \mathbf{x} . Then $|\mathcal{C}| \leq 2m^{D-1}$.*

Proof. To begin, choose a set $S \subset [D]$ with $|S| = d$ and a vector $\mathbf{z} \in [m]^{[D] \setminus S}$ uniformly at random. For each $i \in [m]$ let $\mathbf{z}_S(i) \in [m]^D$ denote the vector which agrees with \mathbf{z} on coordinates in $[D] \setminus S$ and equals i everywhere else. Also let $\mathcal{B}_{S,\mathbf{z}}$ denote the combinatorial line $\mathcal{B}_{S,\mathbf{z}} := \{\mathbf{z}_S(i) : i \in [m]\}$.

Now as \mathcal{C} does not contain any d -dominating pairs, for any choice of S and \mathbf{z} we have $|\mathcal{C} \cap \mathcal{B}_{S,\mathbf{z}}| \leq 1$. Letting X_i denote the indicator random variable which is 1 if $\mathbf{z}_S(i) \in \mathcal{C}$ and 0 otherwise, this gives

$$\sum_{i \in [m]} X_i \leq 1.$$

Taking expectations over all choice of S and \mathbf{z} , this gives

$$\sum_{C \in \mathcal{C}} \mathbb{P}(C \in \mathcal{B}_{S,\mathbf{z}}) = \sum_{i \in [m]} \sum_{C \in \mathcal{C}} \mathbb{P}(\mathbf{z}_S(i) = C) \leq 1. \quad (16)$$

However, an easy calculation gives that if C has k_i entries i for all $i \in [m]$, then

$$\mathbb{P}(C \in \mathcal{B}_{S,\mathbf{z}}) = \sum_{i \in [m]} \frac{\binom{k_i}{d}}{m^{D-d} \binom{D}{d}}.$$

This expression is minimized when all k_i are as equal as possible. Thus

$$\begin{aligned} \mathbb{P}(C \in \mathcal{B}_{S,\mathbf{z}}) &\geq m \frac{\binom{D/m}{d}}{m^{D-d} \binom{D}{d}} = m \frac{(D/m)_d}{m^{D-d} D_d} = \frac{m}{m^D} \prod_{l \in [0, d-1]} \left(1 - \frac{l(m-1)}{D-l}\right) \\ &\geq \frac{1}{m^{D-1}} \left(1 - \sum_{l \in [0, d-1]} \frac{l(m-1)}{D/2}\right) \\ &\geq \frac{1}{m^{D-1}} \left(1 - \frac{md^2}{D}\right) \geq \frac{1}{2m^{D-1}}. \end{aligned}$$

The final line here used $2md^2 \leq D$. Combined with (16) this gives $|\mathcal{C}|/2m^{D-1} \leq 1$, as required. \square

We are now ready for the proof of Theorem 3.

Proof of Theorem 3. By Proposition 2.1 it suffices to prove the theorem for $n = 2k$. Let $m = \lfloor \frac{\log_2(n/d^2)}{2} \rfloor$. For convenience we assume that n is divisible by m , with $Km = n$. Let $[n] = \bigcup_{i=1}^K I_i$ be a partition of $[n]$ where $I_i = \{(i-1)m+1, \dots, im\}$ for all $i \in [K]$. Given a set $T \subset [K]$, let $T^c = [K] \setminus T$ and let

$$\mathcal{B}_T := \{A \subset \bigcup_{i \in T^c} I_i : |A \cap I_i| \neq 1 \text{ for all } i \in T^c\}.$$

Given $B \in \mathcal{B}_T$ and $\mathbf{x} \in [m]^T$ we also let $B(\mathbf{x}) := B \cup \{(i-1)m+j-1 : i \in T, x_i = j\}$ and

$$\mathcal{C}_B := \{B(\mathbf{x}) : \mathbf{x} \in [m]^T\}.$$

Note that for every $A \subset [n]$ there is a unique $T \subset [K]$, $B \in \mathcal{B}_T$ and $\mathbf{x} \in [m]^T$ such that $A = B(\mathbf{x})$. Thus we have the disjoint union

$$\binom{[n]}{n/2} = \bigcup_{T \subset [K]} \bigcup_{\substack{B \in \mathcal{B}_T \\ |B| = \frac{n}{2} - |T|}} \mathcal{C}_B. \quad (17)$$

We will first show that almost all sets A in $\binom{[n]}{n/2}$ are of the form $A = B(\mathbf{x})$ where $T \subset [K]$ and $B \in \mathcal{B}_T$ with $|T| \geq mK/2^{m+1} = n/2^{m+1}$. To see this, given a set $A \subset [n]$, let $A_i = A \cap I_i$ for all $i \in [K]$. We will say that $A \subset [n]$ is *bad* if $T(A) = \{i \in [K] : |A_i| = 1\}$ satisfies $|T(A)| \leq \frac{m}{2^{m+1}} K$. We claim that there are at most $O(e^{-n^{1/2}/2} 2^n)$ sets are bad. Indeed, if we select $A \subset [n]$ uniformly at random, we have $\mathbb{P}(|A_i| = 1) = m/2^m$, which gives $\mathbb{E}(|T(A)|) = \frac{mK}{2^m} = \frac{n}{2^m}$. As $|A_i| = 1$ for each $i \in [K]$ independently, by Chernoff's inequality, we find that $\mathbb{P}(|T(A)| - \frac{n}{2^m} \leq -\frac{n}{2^{m+1}}) \leq e^{-\frac{n}{2^{m+1}}}$. As $m \leq \log_2(n/d^2)/2 \leq \frac{\log_2 n}{2}$ we find that $\mathbb{P}(A \text{ is bad}) \leq e^{-n^{1/2}/2}$. Equivalently, $|\{A \subset [n] : A \text{ is bad}\}| = O(e^{-n^{1/2}/2} 2^n)$.

Now suppose that $T \subset [K]$ with $|T| \geq n/2^{m+1}$ and $B \in \mathcal{B}_T$. Note that given $\mathbf{x}, \mathbf{y} \in [m]^T$, if \mathbf{y} d -dominates \mathbf{x} then $\text{pat}(B(\mathbf{x}), B(\mathbf{y})) = \text{AP}(d)$. Noting that as $m = \lfloor \log_2(n/d^2)/2 \rfloor$ we have $|T| \geq n/2^{m+1} \geq 2^m d^2 \geq 2md^2$. Setting $D = |T|$, Lemma 4.1 therefore shows that any $\mathcal{A} \subset \binom{[n]}{n/2}$ which is $\text{AP}(d)$ -free satisfies

$$|\mathcal{A} \cap \mathcal{C}_B| \leq 2m^{|T|-1} = \frac{2}{m} |\mathcal{C}_B|. \quad (18)$$

Summing over all $T \subset [K]$ and $B \in \mathcal{C}_T$, combined with (17) and (18), this gives

$$\begin{aligned} |\mathcal{A}| &\leq \sum_{T \subset [K]} |\mathcal{A} \cap \bigcup_{\substack{B \in \mathcal{B}_T \\ |B|=n/2-|T|}} \mathcal{C}_B| \leq |\{A \subset [n] : A \text{ bad}\}| + \sum_{\substack{T \subset [K]: \\ |T| \geq 2md^2}} \sum_{B \in \mathcal{B}_T} |\mathcal{A} \cap \mathcal{C}_B| \\ &\leq O\left(\frac{2^n}{e^{\sqrt{n}/2}}\right) + \sum_{\substack{T \subset [K]: \\ |T| \geq 2md^2}} \sum_{\substack{B \in \mathcal{B}_T \\ |B|=n/2-|T|}} \frac{2}{m} |\mathcal{C}_B| \\ &\leq \frac{2+o(1)}{m} \binom{n}{n/2}. \end{aligned}$$

This completes the proof of the theorem. \square

5 Concluding remarks and open problems

In this paper we proved bounds on the size of families $\mathcal{A} \subset \mathcal{P}[n]$ which avoid a d -balanced pattern P . Our proof shows that such families satisfy

$$|\mathcal{A}| = O(a_d n^{-c_d} 2^n),$$

where $a_d = (8d)^{5d}$ and $c_d = 6d8^{-d}$. In particular, families \mathcal{A} which avoid a d -balanced pattern with $d < c \log \log n$ satisfy $|\mathcal{A}| = o(2^n)$ for some absolute constant $c > 0$. It would be interesting to improve the density bound here and/or extend the range of d for which this zero density property holds.

Another interesting question is the following: which balanced pattern P has the strongest effect on the density of P -free families $\mathcal{A} \subset \mathcal{P}[n]$? That is, what is $\min_P \delta(n, k, P)$, where the minimum is taken over all balanced patterns P ? If instead of patterns we only forbid intersection sizes (as discussed in the Introduction) then there are a number of very strong density results for subsets of $\mathcal{P}[n]$. For example, the Frankl-Rödl [10] theorem shows that given $\epsilon > 0$, if $\mathcal{A} \subset \mathcal{P}[n]$ and $|A \cap B| \neq t$ for some $\epsilon n \leq t \leq (1/2 - \epsilon)n$ then $|\mathcal{A}| \leq (2 - \delta)^n$, where $\epsilon = \epsilon(\delta) > 0$. It would be very interesting to know if there exists a pattern which forces a superpolynomial density in n . That is, does there an increasing sequence of naturals $(n_k)_{k \in \mathbb{N}}$ and balanced patterns $(P_k)_k$ with $\delta(n_k, n_k/2, P_k) = n_k^{-\omega_k(1)}$ for some function $\omega_k(1)$ tending to infinity with k ?

Lastly, how large can d be (as a function of n) while still giving $\delta(n, n/2, \text{AP}(d)) \rightarrow 0$ as $n \rightarrow \infty$. Theorem 3 proves that this holds for any $d = o(\sqrt{n})$.

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